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Research on Fractional Integral Problems of Two Fractional Rational Functions

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Abstract: **In this paper, based on Jumarie's modified Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, we study fractional integral problems of two fractional rational functions. Using some methods, the exact solutions of these two fractional integrals can be obtained. Moreover, our results are generalizations of classical calculus results.**

Keywords: **Jumarie's modified R-L fractional calculus, new multiplication, fractional analytic functions, fractional integral, fractional rational functions.**

I. INTRODUCTION

Fractional calculus is a natural extension of classical calculus, which has a history of more than 300 years. In fact, since the birth of differential and integral theory, several mathematicians have studied their ideas on the calculation of non-integer order derivatives and integrals. However, although much work has been done, the application of fractional derivatives and integrals has only recently begun. In recent years, the development of fractional calculus has stimulated people's new interest in physics, engineering, economics, biology, control theory, and other fields [1-12].

However, fractional calculus is different from traditional calculus. The definition of fractional derivative is not unique. Common definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [13-17]. Because Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with traditional calculus.

In this paper, based on Jumarie type of R-L fractional calculus and a new multiplication of fractional analytic functions, we study the following fractional integral problems of two α-fractional rational functions:

$$\left({}_{0}I_{x}^{\alpha}\right)\left[\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_{\alpha}3}\right]^{\otimes_{\alpha}(-1)}\right],$$

$$\left({}_{0}I_{x}^{\alpha}\right)\left[\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_{\alpha}6}\right]^{\otimes_{\alpha}(-1)}\right],$$

where $0<\alpha\leq 1$. Using some techniques, the exact solutions of these two α-fractional integrals can be obtained. In fact, our results are generalizations of ordinary calculus results.

II. PRELIMINARIES

Firstly, we introduce the fractional calculus used in this paper.

Definition 2.1 ([18])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \tag{1}$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \tag{2}$$

where $\Gamma(\ \ )$ is the gamma function.

In the following, some properties of Jumarie type of R-L fractional derivative are introduced.

Proposition 2.2 ([19])**:** *If* α,β,x_0,c *are real numbers and* $\beta\geq\alpha>0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right]=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[c]=0. \tag{4}$$

Next, we introduce the definition of fractional analytic function.

Definition 2.3 ([20])**:** If x,x_0, and a_k are real numbers for all k, $x_0\in(a,b)$, and $0<\alpha\leq 1$. If the function $f_\alpha:[a,b]\to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha:[a,b]\to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([21])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{5}$$

$$g_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\ . \tag{6}$$

Then we define

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)\\&=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\otimes_\alpha\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\\&=\sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(x-x_0)^{n\alpha}.\end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)\\&=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\otimes_\alpha\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\\&=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}.\end{aligned} \tag{8}$$

Definition 2.5 ([22])**:** If $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x - x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x - x_0)^\alpha\right)^{\otimes_\alpha n}, \tag{9}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x - x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x - x_0)^\alpha\right)^{\otimes_\alpha n}. \tag{10}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha(g_\alpha(x^\alpha)) = \sum_{n=0}^{\infty} \frac{a_n}{n!}(g_\alpha(x^\alpha))^{\otimes_\alpha n}, \tag{11}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha(f_\alpha(x^\alpha)) = \sum_{n=0}^{\infty} \frac{b_n}{n!}(f_\alpha(x^\alpha))^{\otimes_\alpha n}. \tag{12}$$

Definition 2.6 ([23])**:** If $0 < \alpha \leq 1$, and x is a real variable. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}. \tag{13}$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$. On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^k x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \tag{14}$$

and

$$sin_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{15}$$

Definition 2.7 ([24])**:** Let $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $(f_\alpha(x^\alpha))^{\otimes_\alpha n} = f_\alpha(x^\alpha) \otimes_\alpha \cdots \otimes_\alpha f_\alpha(x^\alpha)$ is called the nth power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) = 1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $(f_\alpha(x^\alpha))^{\otimes_\alpha (-1)}$.

III. MAIN RESULTS

In this section, we solve two fractional integrals of fractional rational functions.

Theorem 3.1: *Let* $0 < \alpha \leq 1$, *then*

$$\left({}_0I_x^\alpha\right)\left[\left[1 + \left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 3}\right]^{\otimes_\alpha (-1)}\right]$$

$$= \frac{1}{3} \cdot Ln_\alpha\left(\left|\frac{1}{\Gamma(\alpha+1)}x^\alpha + 1\right|\right) - \frac{1}{6} \cdot Ln_\alpha\left(\left|\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2} - \frac{1}{\Gamma(\alpha+1)}x^\alpha + 1\right|\right)$$

$$+ \frac{1}{\sqrt{3}} \cdot arctan_\alpha\left(\frac{1}{\sqrt{3}}\left[2 \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha - 1\right]\right) + \frac{1}{\sqrt{3}} \cdot arctan_\alpha\left(\frac{1}{\sqrt{3}}\right). \tag{16}$$

Proof $\left({}_0I_x^\alpha\right)\left[\left[1 + \left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 3}\right]^{\otimes_\alpha (-1)}\right]$

$$= \left({}_0I_x^\alpha\right)\left[\frac{1}{3} \cdot \left[\frac{1}{\Gamma(\alpha+1)}x^\alpha + 1\right]^{\otimes_\alpha (-1)} + \left(-\frac{1}{3} \cdot \frac{1}{\Gamma(\alpha+1)}x^\alpha + \frac{2}{3}\right) \otimes_\alpha \left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2} - \frac{1}{\Gamma(\alpha+1)}x^\alpha + 1\right]^{\otimes_\alpha (-1)}\right]$$

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$$=\frac{1}{3}\cdot\left({}_0I_x^{\alpha}\right)\left[\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right]^{\otimes_\alpha(-1)}\right]-\frac{1}{3}\cdot\left({}_0I_x^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-2\right)\otimes_\alpha\left[\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2}-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right]^{\otimes_\alpha(-1)}\right]. \quad (17)$$

Since

$$\frac{1}{3}\cdot\left({}_0I_x^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-2\right)\otimes_\alpha\left[\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2}-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right]^{\otimes_\alpha(-1)}\right]$$

$$=\frac{1}{6}\cdot\left({}_0I_x^{\alpha}\right)\left[\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-4\right)\otimes_\alpha\left[\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2}-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right]^{\otimes_\alpha(-1)}\right]$$

$$=\frac{1}{6}\cdot\left({}_0I_x^{\alpha}\right)\left[\left(2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-1\right)\otimes_\alpha\left[\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2}-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right]^{\otimes_\alpha(-1)}\right]$$

$$-\frac{1}{2}\cdot\left({}_0I_x^{\alpha}\right)\left[\left[\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2}-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right]^{\otimes_\alpha(-1)}\right]$$

$$=\frac{1}{6}\cdot\left({}_0I_x^{\alpha}\right)\left[\left[\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2}-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right]^{\otimes_\alpha(-1)}\otimes_\alpha\left({}_0D_x^{\alpha}\right)\left[\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2}-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right]\right]$$

$$-\frac{1}{2}\cdot\left({}_0I_x^{\alpha}\right)\left[\left[\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-\frac{1}{2}\right]^{\otimes_\alpha 2}+\frac{3}{4}\right]^{\otimes_\alpha(-1)}\right]$$

$$=\frac{1}{6}\cdot Ln_\alpha\left(\left|\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2}-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right|\right)-\frac{1}{\sqrt{3}}\cdot arctan_\alpha\left(\frac{1}{\sqrt{3}}\left[2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-1\right]\right)-\frac{1}{\sqrt{3}}\cdot arctan_\alpha\left(\frac{1}{\sqrt{3}}\right). \quad (18)$$

It follows that

$$\left({}_0I_x^{\alpha}\right)\left[\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 3}\right]^{\otimes_\alpha(-1)}\right]$$

$$=\frac{1}{3}\cdot Ln_\alpha\left(\left|\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right|\right)-\frac{1}{6}\cdot Ln_\alpha\left(\left|\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2}-\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right|\right)$$

$$+\frac{1}{\sqrt{3}}\cdot arctan_\alpha\left(\frac{1}{\sqrt{3}}\left[2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-1\right]\right)+\frac{1}{\sqrt{3}}\cdot arctan_\alpha\left(\frac{1}{\sqrt{3}}\right).$$

Q.e.d.

Theorem 3.2: *If* $0<\alpha\le 1$, *then*

$$\left({}_0I_x^{\alpha}\right)\left[\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 6}\right]^{\otimes_\alpha(-1)}\right]$$

$$=\frac{1}{2}\cdot arctan_\alpha(x^{\alpha})+\frac{1}{6}\cdot arctan_\alpha\left(\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 3}\right)$$

$$-\frac{1}{4\sqrt{3}}\cdot Ln_\alpha\left(\left|\left[\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2}-\sqrt{3}\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right]\otimes_\alpha\left[\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]^{\otimes_\alpha 2}+\sqrt{3}\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+1\right]^{\otimes_\alpha(-1)}\right|\right). \quad (19)$$

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Proof $\left({}_0I_x^\alpha\right)\left[\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 6}\right]^{\otimes_\alpha(-1)}\right]$

$=\frac{1}{2}\left({}_0I_x^\alpha\right)\left[2\cdot\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 6}\right]^{\otimes_\alpha(-1)}\right]$

$=\frac{1}{2}\left({}_0I_x^\alpha\right)\left[\left[1-\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 4}+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+1-\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 4}\right]\otimes_\alpha\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 6}\right]^{\otimes_\alpha(-1)}\right]$

$=\frac{1}{2}\left({}_0I_x^\alpha\right)\left[\left[1-\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 4}\right]\otimes_\alpha\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 6}\right]^{\otimes_\alpha(-1)}\right]$

$+\frac{1}{2}\left({}_0I_x^\alpha\right)\left[\left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}\right]\otimes_\alpha\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 6}\right]^{\otimes_\alpha(-1)}\right]+\frac{1}{2}\left({}_0I_x^\alpha\right)\left[\left[1-\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 4}\right]\otimes_\alpha\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 6}\right]^{\otimes_\alpha(-1)}\right].$

(20)

Since

$$\frac{1}{2}\left({}_0I_x^\alpha\right)\left[\left[1-\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 4}\right]\otimes_\alpha\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 6}\right]^{\otimes_\alpha(-1)}\right]$$

$$=\frac{1}{2}\left({}_0I_x^\alpha\right)\left[\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}\right]^{\otimes_\alpha(-1)}\right]$$

$$=\frac{1}{2}\cdot arctan_\alpha(x^\alpha)\,. \tag{21}$$

And

$$\frac{1}{2}\left({}_0I_x^\alpha\right)\left[\left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}\right]\otimes_\alpha\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 6}\right]^{\otimes_\alpha(-1)}\right]$$

$$=\frac{1}{2}\left({}_0I_x^\alpha\right)\left[\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 6}\right]^{\otimes_\alpha(-1)}\otimes_\alpha\left({}_0D_x^\alpha\right)\left[\frac{1}{3}\cdot\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 3}\right]\right]$$

$$=\frac{1}{6}\cdot arctan_\alpha\left(\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 3}\right). \tag{22}$$

And

$$\frac{1}{2}\left({}_0I_x^\alpha\right)\left[\left[1-\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 4}\right]\otimes_\alpha\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 6}\right]^{\otimes_\alpha(-1)}\right]$$

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$$=\frac{1}{2}\left({}_0I_x^\alpha\right)\left[\left[1-\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}\right]\otimes_\alpha\left[\left[1-\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 4}\right]\right]^{\otimes_\alpha(-1)}\right]$$

$$=\frac{1}{2}\left({}_0I_x^\alpha\right)\left[\left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha(-2)}-1\right]\otimes_\alpha\left[\left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha(-2)}-1+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}\right]\right]^{\otimes_\alpha(-1)}\right]$$

$$=-\frac{1}{2}\left({}_0I_x^\alpha\right)\left[\left[1-\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha(-2)}\right]\otimes_\alpha\left[\left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha(-1)}\right]^{\otimes_\alpha 2}\right]-3\right]^{\otimes_\alpha(-1)}\right]$$

$$=-\frac{1}{2}\left({}_0I_x^\alpha\right)\left[\left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha(-1)}\right]^{\otimes_\alpha 2}-3\right]^{\otimes_\alpha(-1)}\otimes_\alpha\left({}_0D_x^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha(-1)}\right]\right]$$

$$=-\frac{1}{4\sqrt{3}}\cdot Ln_\alpha\left(\left|\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha(-1)}-\sqrt{3}\right]\otimes_\alpha\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha(-1)}+\sqrt{3}\right]^{\otimes_\alpha(-1)}\right|\right)$$

$$=-\frac{1}{4\sqrt{3}}\cdot Ln_\alpha\left(\left|\left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}-\sqrt{3}\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha+1\right]\otimes_\alpha\left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+\sqrt{3}\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha+1\right]^{\otimes_\alpha(-1)}\right|\right). \tag{23}$$

It follows that

$$\left({}_0I_x^\alpha\right)\left[\left[1+\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 6}\right]^{\otimes_\alpha(-1)}\right]$$

$$=\frac{1}{2}\cdot arctan_\alpha(x^\alpha)+\frac{1}{6}\cdot arctan_\alpha\left(\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 3}\right)$$

$$-\frac{1}{4\sqrt{3}}\cdot Ln_\alpha\left(\left|\left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}-\sqrt{3}\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha+1\right]\otimes_\alpha\left[\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]^{\otimes_\alpha 2}+\sqrt{3}\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha+1\right]^{\otimes_\alpha(-1)}\right|\right).$$

Q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie type of R-L fractional calculus and a new multiplication of fractional analytic functions, we solve two fractional integrals of two fractional rational functions by using some techniques. Furthermore, our results are generalizations of traditional calculus results. In the future, we will continue to use Jumarie's modified R-L fractional calculus and the new multiplication of fractional analytic functions to solve the problems in fractional differential equations and engineering mathematics.

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